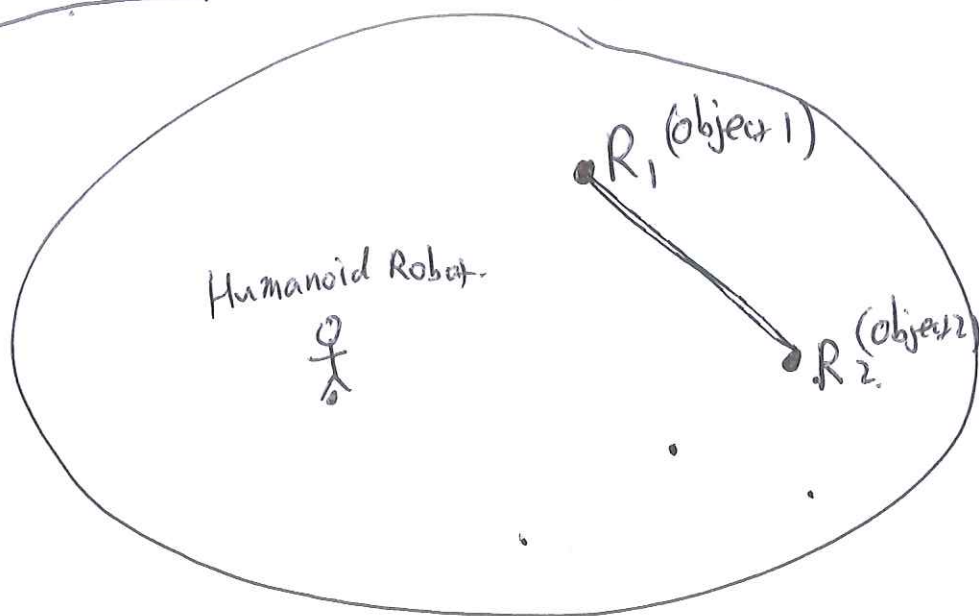


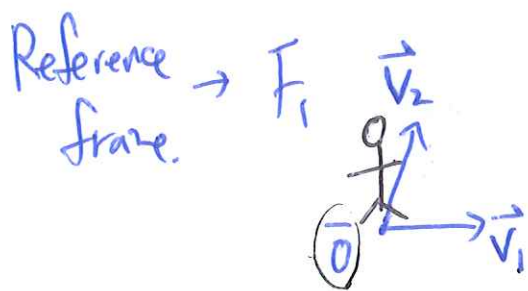
Lec 1 Linear Algebra Review (Representation Theory)



We want to pick up object 1 and place on top of object 2.

Q. How can we identify the position of the robot?

A. We need to put coordinates w.r.t. some reference frame.



1) Choose origin.

2) Choose fundamental vectors $(\vec{v}_1, \vec{v}_2)$

In the reference frame F_1 ,

$$R_1 = 2 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 \quad \xrightarrow{\text{Representation w.r.t. } F_1} R_1^F = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$R_2 = 4 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \quad \xrightarrow{\text{Representation w.r.t. } F_1} R_2^F = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Any object R on a plane $\exists d_1, d_2 \in \mathbb{R}$

↑
there exist

In this case $\mathbb{R}^2$

s.t. $R = d_1 \vec{v}_1 + d_2 \vec{v}_2$

↑
such that

All collection of these points form "Linear Vector Space"

Def Linear Space over a field $(\mathbb{R})$ $V(X, +, \mathbb{R}, \cdot, \cdot)$

↑
Vector Space

For every $\vec{v}, \vec{u}, \vec{w} \in X$ and $a, b \in \mathbb{R}$

o) Closed under $+$, $\cdot$: $\vec{u} + \vec{w} \in X$ and $a \cdot \vec{u} \in X$

1) Associativity : $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

2) Commutativity : $\vec{u} + \vec{w} = \vec{w} + \vec{u}$

3) identity $(+)$: $\exists \vec{0} \in X$ s.t. $\vec{u} + \vec{0} = \vec{u}$

4) inverse $(+)$: $\exists \vec{q} \in X$ s.t. $\vec{u} + \vec{q} = \vec{0}$

5) Multiplicative : $a \cdot (b \cdot \vec{u}) = (a \cdot b) \cdot \vec{u}$

6) Identity over field : $1 \cdot \vec{u} = \vec{u}$

7) Distributive $(+)$: $d \cdot (\vec{u} + \vec{v}) = d \cdot \vec{u} + d \cdot \vec{v}$

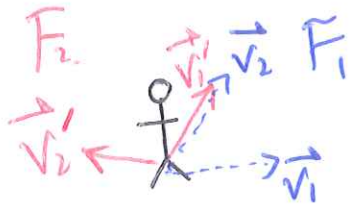
8) Distributive $(\cdot)$: $(a+b) \cdot \vec{u} = a \cdot \vec{u} + b \cdot \vec{u}$

Example : $\mathbb{R}^2$ planar $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$\mathbb{R}^3$ Spatial $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

Q. Can we choose any reference frame $\vec{v}_1, \vec{v}_2$?

Example 1



• R_1

• R_2

$$R_1 = 2 \cdot \vec{v}_1' + (-2) \cdot \vec{v}_2'$$

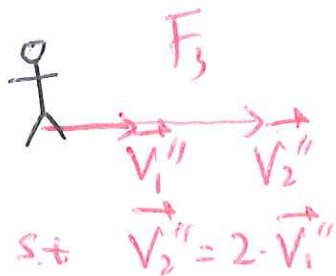
$$R_2 = 0 \cdot \vec{v}_1' + (-4) \cdot \vec{v}_2'$$

$$R_1^{F_2} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$R_2^{F_2} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

Example 2

• R_1



then

• R_2

$$R_2 = 4 \cdot \vec{v}_1'' = 2 \cdot (2\vec{v}_1'') = 2 \cdot \vec{v}_2''$$

We can't represent R_1 in F_3

$$\hookrightarrow -2 \cdot \vec{v}_1'' + 1 \cdot \vec{v}_2'' = 0$$

Def (Linearly Independent)

If only the trivial combination gives zero, so that

$$C_1 \cdot \vec{v}_1 + C_2 \cdot \vec{v}_2 + \dots + C_k \cdot \vec{v}_k = \vec{0} \text{ where } C_1, \dots, C_k \in \mathbb{R}$$

only happens when $C_1 = C_2 = \dots = C_k = 0$

$\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.

Def | (Span of $\{\vec{v}_1, \dots, \vec{v}_k\}$)

If any element $\vec{x} \in X$, there exist a linear combination s.t $\vec{x} = C_1 \vec{v}_1 + \dots + C_k \vec{v}_k$ for some

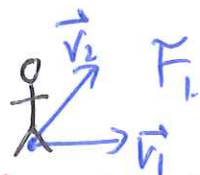
$C_1, \dots, C_k \in \mathbb{R}$, then these vectors "Span" the space

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = X$$

Q. So what will be a "good" reference frame?

A. 1) The fundamental vectors that can "Span" the whole space

2) No redundancy on choosing the representative coordinates.



→ Basis of the linear space.

Def | Basis of the linear vector space X .

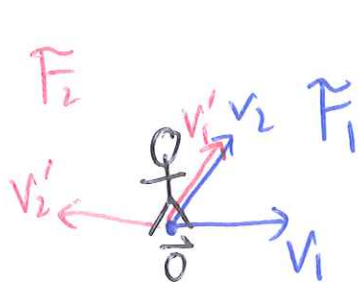
A set of vectors is called a basis for a vector space

if 1) It is linearly independent

2) It spans the whole space

Representation Theory

Back to the example.



• R_1

• R_2

Recall

$$R_1^{F_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$R_2^{F_2} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Recall

$$R_1^{F_2} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$R_2^{F_1} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

X is a vector space $\mathbb{R}^2$
collection of all vectors
from the origin $\vec{0}$.

Q. Did the object R_1, R_2 moved?

No! The coordinate is a "representation" of an object
associated to the frame you are observing!

Q. How can we change the representation of one frame
to the other, knowing the relation between F_1 & F_2



Observe that.

$$\vec{V}_1' = \vec{V}_2$$

$$\vec{V}_2' = -\vec{V}_1$$

Recall that the position of R_1 and R_2 relative to the origin did not change.

$$R_1 = 2 \cdot \vec{V}_1' - 2 \cdot \vec{V}_2' = 2 \cdot (\vec{V}_2) - 2 \cdot (-\vec{V}_1) = 2 \cdot \vec{V}_1 + 2 \cdot \vec{V}_2$$

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} \text{ in } F_2 \longrightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ in } F_1$$

Generalization Any coordinate $\begin{bmatrix} a \\ b \end{bmatrix}$ in F_2 represent some R .

$$R = a \cdot \vec{V}_1' + b \cdot \vec{V}_2' = a \cdot \vec{V}_2 + b \cdot (-\vec{V}_1) = -b \vec{V}_1 + a \vec{V}_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \text{ in } F_2 \longrightarrow \begin{bmatrix} -b \\ a \end{bmatrix} \text{ in } F_1$$

Q. Is there a matrix which can convert coordinate transformation?

$$\begin{bmatrix} -b \\ a \end{bmatrix} = A \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

First column

coordinates of

$\vec{V}_1'$ in F_1

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ in } F_1$$

$$\vec{V}_1' = \vec{V}_2$$

$$= 0 \cdot \vec{V}_1 + 1 \cdot \vec{V}_2$$

second column

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ in } F_1$$

$$\vec{V}_2' = -\vec{V}_1$$

coordinates of

$\vec{V}_2'$ in F_1 (6)

A is called change of basis transformation